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The Decoupling of Elastic Waves from a Weak Formulation Perspective

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Abstract. Two weak formulations for the Lamé system with the boundary conditions of third and fourth types are proposed. It is shown that the regularity of the solutions and properties of the boundary surface guarantee the equivalence of variational and standard formulations of the problem. Moreover, if the boundary of Ω is a Lipschitz polyhedron or if $\mathscr{S}(\boldsymbol{x}) = 0$ on $\partial\Omega$, the decoupling results of [8] are derived from the weak formulations.

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1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded simply-connected domain with a connected piecewise $C^{2,1}$-smooth boundary $\partial\Omega$. An isotropic elastic medium characterised by the Lamé constants $\lambda > 0$ and $\mu > 0$, occupies Ω. Given a source term $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, we consider the linearised elasticity equation

$$-\Delta^* \boldsymbol{u} - \omega^2 \boldsymbol{u} = \boldsymbol{f} \quad \text{in} \quad \Omega, \tag{1.1}$$

where $\boldsymbol{u} = [u_j(\boldsymbol{x})]_{j=1}^3$ is the displacement field, $\omega > 0$ the angular wavenumber, and the operator Δ^* is defined by

$$\Delta^* \boldsymbol{u} := \mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}) = -\mu\nabla\wedge(\nabla\wedge\boldsymbol{u}) + (\lambda+2\mu)\nabla(\nabla\cdot\boldsymbol{u}), \tag{1.2}$$

where $\wedge$ is the cross product of two vector fields.

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The Lamé system (1.1) is complemented by the boundary conditions of different kinds [6,7]. The first kind boundary condition has the form

$$\boldsymbol{u} = \mathbf{0} \quad \text{on} \quad \partial\Omega.$$

The second one reads

$$\boldsymbol{T}\boldsymbol{u} = \mathbf{0} \quad \text{on} \quad \partial\Omega,$$

with the traction operator $\mathbf{T}$,

$$\boldsymbol{T}\boldsymbol{u} := \lambda(\boldsymbol{\nabla}\cdot\boldsymbol{u})\boldsymbol{\nu} + \mu(\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}\boldsymbol{u}^T)\boldsymbol{\nu} = 2\mu\partial_{\boldsymbol{\nu}}\boldsymbol{u} + \lambda\,\boldsymbol{\nu}\boldsymbol{\nabla}\cdot\boldsymbol{u} + \mu\,\boldsymbol{\nu}\wedge(\boldsymbol{\nabla}\wedge\boldsymbol{u}),$$

where T is the transposition operation, $\boldsymbol{\nu}\in\mathbb{S}^2$ the outward unit normal vector to $\partial\Omega$, and $\partial_{\boldsymbol{\nu}}$ a boundary differential operator defined by

$$\partial_{\boldsymbol{\nu}}\boldsymbol{u} := [\,\boldsymbol{\nu}\cdot\boldsymbol{\nabla}u_j]_{j=1}^3.$$

The equations

$$\boldsymbol{\nu}\cdot\boldsymbol{u} = 0, \quad \boldsymbol{\nu}\wedge\boldsymbol{T}\boldsymbol{u} = \mathbf{0} \quad \text{on} \quad \partial\Omega, \tag{1.3}$$

and

$$\boldsymbol{\nu}\wedge\boldsymbol{u} = \mathbf{0}, \quad \boldsymbol{\nu}\cdot\boldsymbol{T}\boldsymbol{u} = 0 \quad \text{on} \quad \partial\Omega \tag{1.4}$$

represent the boundary conditions of the third and fourth kind, respectively.

Here, we mainly focus on the Lamé system (1.1) provided with the boundary condition (1.3) or (1.4). It is known that the elastic body waves can be decomposed into two parts — viz. pressure and shear waves. These waves generally coexist and simultaneously propagate with different speeds and in different directions. In recent work [8], a complete decoupling of the pressure and shear waves is established for the third and fourth boundary conditions under geometric conditions on the boundary of an impenetrable scatterer. This allows to reformulate the Lamé system as Helmholtz and Maxwell systems.

The goal of this work is to revisit the decoupling results [8] in order to obtain new relevant variational formulations and to show the well-posedness of the corresponding variational problems. We observe that the problems (3.5) and (3.11) are well-posed without any geometric assumption concerning the boundary surface. Furthermore, under an assumption about the regularity of the solution, the variational formulations are rewritten as Helmholtz and Maxwell systems. If the boundary satisfies additional geometric conditions, the variational formulations and the original Lamé system are equivalent. This means that the decoupling results deduced from the Lamé system in [8] are retrieved from a weak formulation perspective.

To the best of our knowledge, the variational formulations (3.5) and (3.11) are novel and they can find numerous applications in numerical simulation of the Lamé system with the boundary conditions (1.3) and (1.4), including the variational discretisation such as finite element methods [2], [3], [5], [9]. Due to Theorems 3.3 and 3.6, computations and the related numerical analysis may be performed separately on the Helmholtz equation and the Maxwell system.